EMC 2008 19th INTERNATIONAL WROCLAW SYMPOSIUM AND EXHIBITION ON ELECTROMAGNETIC COMPATIBILITY, WROCLAW, 11 – 13 JUNE, 2008
www.emc.wroc.pl

METHODS OF CALCULATING THE MEAN AND VARIANCE OF POWER SUMS WITH LOG-NORMAL COMPONENTS

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Abstract: Different methods of calculation the mean and variance of power sums with log-normal components of the field strength in the planning and compatibility of radio systems analysis were described. Samples of the numerical calculations of the coverage areas for DVB-T Single Frequency Network (SFN) were performed. Practical scope for each method was discussed.

Keywords: Log-Normal Distribution, Mean, Variance, Power Sum, SFN, DVB-T.

1. INTRODUCTION

The log-normal distribution of the received signals arises in many radiocommunication areas, such as radio network planning including electromagnetic compatibility of the cellular mobile systems, wide band transmissions, and new digital broadcasting systems, eg. mobile radio and television.

The attenuation due to shadowing and multipath propagation in wireless channels is often modeled by the log-normal distribution (LND). LND characterizes the co-channel interference power from the transmissions in neighboring cells, and is also of interest in outage probability analysis of the wanted signals. The summation of signals has particular significance in a single frequency network (SFN) [1,2] where it is necessary to calculate the sum of the wanted and interfered signals, which have different median values of the log-normal distributions. Therefore, one often encounters the sum of log-normal random variables (RV) in analyzing of the radio systems performances.

Given the importance of the sum of log-normal distributions, considerable efforts have to be devoted to analyze its statistical properties. While exact expressions of the log-normal sum probability distribution function are unknown, several analytical approximation methods exist in the literature [2], [3], [5]-[9].

Calculation of statistical properties up to the second order (the first moment = mean, and the second central moment = variance) of the sum of log-normal variables, is not a trivial process.

When implementing numerically, in most cases, it is iterative task, where dealing of differences of large numbers or multiplication of very large and very small components, which can lead to numerical instability.

2. THEORY

In many areas radiocommunication engineering, logarithms of sums of powers are considered in the form

$$P_K = 10\log_{10}\left[\sum_{k=1}^{K} 10^{\frac{X_k}{10}}\right] \tag{1}$$

where $X_1,\ldots, X_K$ are random Gaussian variables.

If X_k is Gaussian random variable, then quantity

$$L_k = 10^{\frac{X_k}{10}} \tag{2}$$

is random variable (RV) with log-normal distribution.

The power sum , as defined in (1), is expressed in decibels. However, in case of using the definition for L_k given in (2), it is more convenient to use the natural logarithm

$$L_k = e^{Y_k} \tag{3}$$

The relationship between the two associated normal variables can be find very easy by comparing right-hands of equations (2) and (3)

$$Y_k = \lambda X_k \tag{4}$$

where

$$\lambda = \frac{1}{10}\log_e 10 = 0.2302585 \tag{5}$$

When the mean and variance of X_k are specified as m_k and σ_k, in order to use the representation (3), we have the obvious scaling

$$\begin{aligned} m_y &= \lambda m_x \\ \sigma_y^2 &= \lambda^2 \sigma_x^2 \end{aligned} \tag{6}$$

Defining the quantities

$$L = 10^{\frac{P_K}{10}} = e^Z \tag{7}$$

the corresponding moments are related by

$$m_Z = \lambda m_{P_K}$$
$$\sigma_Z^2 = \lambda^2 \sigma_{P_K}^2 \quad (8)$$

The nth moment of a log-normal variable L is given in terms of the moment generating function of Y

$$E[L^n] = E[e^{nY}] = e^{nm_Y + \frac{n^2\sigma_Y^2}{2}} \quad (9)$$

The main lemma, when summing log-normal variables, proofed by Marlow [4], says that the sum of log-normal distributed variables is also approximately log-normal.

This lemma and equations are used to compute the first (mean) and the second central moment (variance) of the sum of the log-normal components.

3. IMPLEMENTATIONS

3.1. Wilkinson approximation

The Wilkinson approximation [8] proceeds as follows: Let Y_1 and Y_2 are Gaussian random variables identically distributed. Their sum L is equal

$$L = L_1 + L_2 = e^Z \quad (10)$$

where Z is the associated Gaussian random variable, with mean value m_Z and variance σ_Z and can be expressed as

$$L = e^Z = e^{Y_1} + e^{Y_2} = 2e^Y \quad (11)$$

To find quantities m_Z and σ_Z one uses (9) and equates the first of the two moments of both sides of (11). Taking logarithms gives a set of linear equations for the two unknown values: m_Z and σ_Z.

Note that the moments described by (9) grow exponentially with the order n^2. Dealing with differences of large numbers can lead to numerical instability when trying to evaluate the approximate log-normal density of the sum.

As shown in [8] the Wilkinson approach tends to breaks down for σ_X greater then 4 dB, comparing with the precise Monte Carlo simulation, which is consistent with the above-quoted result from Marlow.

3.2. *k*-Log-Normal Method (*k*-LNM)

The log-normal method (LNM) detailed described in [3] is approximated version of analytical form of median values. To improve the accuracy of the LNM in the high probability region (that is, a high coverage probability) a correction factor can be introduced. This version of the LNM is called *k*-LNM.

k-LNM suffers from the drawback that the appropriate correction factor *k* depends on the number, the powers and the variances of the fields being summed, as well as the location percentage for which the calculation is being done. To obtain optimal results, an interpolation table for the derivation of the value of *k* would be necessary, which is not suitable for a heuristic approach like *k* -LNM. Therefore, to keep the simple and analytic character of the approximation, an average value of *k* is chosen, derived from a sample of representative field configurations. This simplification still results in an inaccuracy for a few dBs, typical configurations which amount to some dBs for 99% locations. For the summation of fields with standard deviations between 6 and 10 dB the value k = 0.5 seems to be a fair compromise. For smaller values of standard deviations a higher value for *k* should be used, e.g. k = 0.7. If *k* is set to 1.0, *k* -LNM is identical to the standard LNM approach. This choice should be especially taken to the summation of interfering fields, since for these the low probability domain is of interest.

3.3. *t*-Log-Normal Method (*t*-LNM)

The *t*-LNM method [2] is a numerical approximation method for the statistical computation of the sum distribution of several log-normally distributed variables. Its structure is similar to that of the standard LNM and it is based on the same idea, i.e. that the sum distribution of two lognormal variables is also log-normal. However, the parameters of the sum distribution are calculated in a different way and, as a consequence, results are different from those of the standard LNM.

This approach leads to a higher accuracy in the high probability region (that is, a high coverage value) compared to the standard LNM and *k*-LNM approaches but this must be paid by higher mathematical complexity. The *t*-LNM method is able to process with different standard deviations of the single fields with a few restrictions. The specific case of noise may be also regarded as an interference signal with a standard deviation of 0 dB.

3.4. Schwartz-Yeh approximation

Method proposed by Schwartz and Yeh [8] approximates analytical form by polynomials. The sum is computed iterative. The mean value and variance of the sum of two independent log-normal random variables are given by the following equations [8]:

$$m_z = m_{y_1} + G_1(\sigma_w, m_w) \quad (12)$$

$$\sigma_z^2 = \sigma_{y_1}^2 - G_1^2(\sigma_w, m_w) - 2\rho^2 G_3(\sigma_w, m_w) + G_2(\sigma_w, m_w) \quad (13)$$

$$\sigma_w^2 = \sigma_{y_2}^2 + \sigma_{y_1}^2 \quad (14)$$

$$m_w = m_{y_2} - m_{y_1} \quad (15)$$

$$\rho = -\sigma_{y_1} / \sigma_w \quad (16)$$

$$\log_{10} G_i(\sigma_w, m_w) = \sum_{j=0}^{J} \sum_{k=0}^{K} A_i(j,k) \sigma_w^{j/2} |m_w|^{k/2} \quad (17)$$

where polynomial coefficients $A_i(j,k)$ were obtained for two region of parameter space: the first region for mean in range $-20 \le m_w \le 0$ and standard deviation in range $0 \leq \sigma_w \leq 15$, and second region in ranges $-40 \leq m_w \leq -20$, $0 \leq \sigma_w \leq 15$ respectively.

These values (mean m_Z and variance σ^2_Z) are used with the next log-normal component to compute moments of the new i-th sum. This process is repeated until all components are summed.

3.5. Analytic form (Safak approximation)

Analytical form presented by Schwartz and Yeh (S-Y) has been expanded by Safak [9], taking into account the correlated components. This method, as S-Y, is also iterative. The mean value and variance of the sum of two log-normal random variables is expressed by the following set of equations

$$m_z = m_{y_1} + G_1(\sigma_w, m_w) \tag{18}$$

$$\sigma_z^2 = \sigma_{y_1}^2 - G_1^2(\sigma_w, m_w) - 2\rho^2 G_2(\sigma_w, m_w) + G_3(\sigma_w, m_w) \tag{19}$$

where

$$\rho = -\frac{\sigma_{y_1}}{\sigma_w} \tag{20}$$

$$m_w = m_{y_2} - m_{y_1} \tag{21}$$

$$\sigma_w^2 = \sigma_{y_2}^2 + \sigma_{y_1}^2 \tag{22}$$

$$G_1(\sigma_w, m_w) = m_w \Phi\left(\frac{m_w}{\sigma_w}\right) + \frac{\sigma_w}{\sqrt{2\pi}} e^{-\frac{m_w^2}{2\sigma_w^2}} + \sum_{k=1}^{\infty} C_k \left(F(\sigma_w, m_w, k) + F(\sigma_w, -m_w, k)\right) \tag{23}$$

$$G_3(\sigma_w, m_w) = \sigma_w{}^2 \sum_{k=0}^{\infty} (-1)^k [F(\sigma_w, m_w, k) + F(\sigma_w, -m_w, k+1)] \tag{24}$$

$$G_2(\sigma_w, m_w) = (m_w^2 + \sigma_w^2)\Phi\left(\frac{m_w}{\sigma_w}\right) + (m_w + \ln 4)\frac{\sigma_w}{\sqrt{2\pi}} e^{-\frac{m_w^2}{2\sigma_w^2}} + 2\sum_{k=1}^{\infty} C_k (m - k\sigma^2) F(\sigma_w, m_w, k) + \sum_{k=2}^{\infty} B_{k-1}[F(\sigma_w, m_w, k) + F(\sigma_w, -m_w, k)] \tag{25}$$

$$F(\sigma_w, -m_w, k) = e^{-km_w + \frac{k^2\sigma_w^2}{2}} \Phi\left(\frac{m - k\sigma_w{}^2}{\sigma_w}\right) \tag{26}$$

$$B_k = \frac{2(-1)^{k+1}}{k+1} \sum_{j=1}^{k} \frac{1}{j} \tag{27}$$

$$C_k = \frac{(-1)^{k+1}}{k} \tag{28}$$

$$\Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{\frac{-t^2}{2}} dt \tag{29}$$

4. NUMERICAL EXAMPLES

Figures presented below shows the most useful methods compared to the precise Monte Carlo simulation.

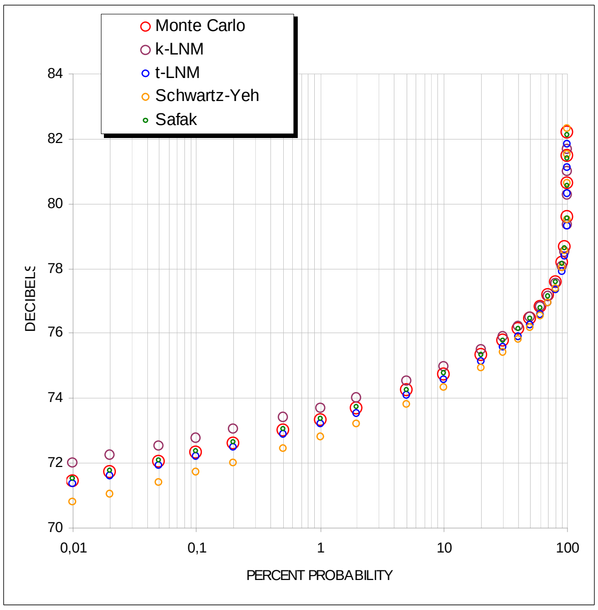

Figure 1. Results of the sum of 24 identically distributed variables with 60 dB means and standard deviations 5 dB.

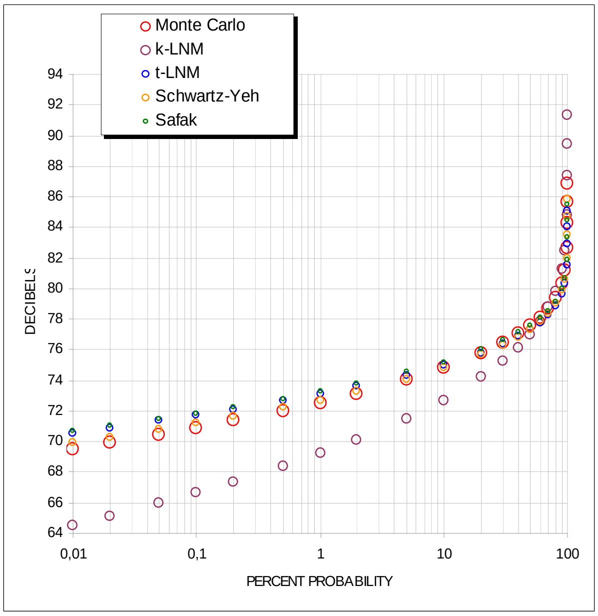

Figure 2. Results of sum of two groups of log-normal variables with 60 dB means and standard deviations as follow:

- Group of 6 RVs with σ = 8 dB,
- Group of 18 RVs with σ = 5 dB.

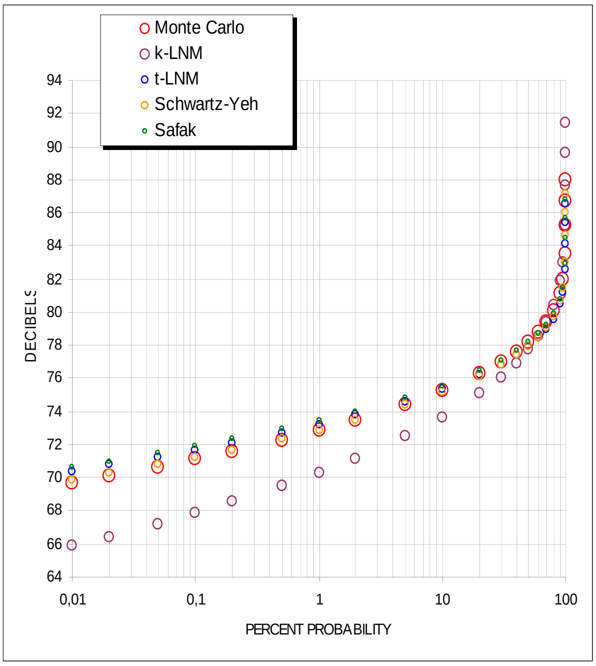

Figure 3. Results of sum of three groups of log-normal variables with this 60 dB means and standard deviations as follow:

- 6 RVs with σ = 8 dB,
- 6 RVs with σ = 7 dB,
- 12 RVs with σ = 5 dB.

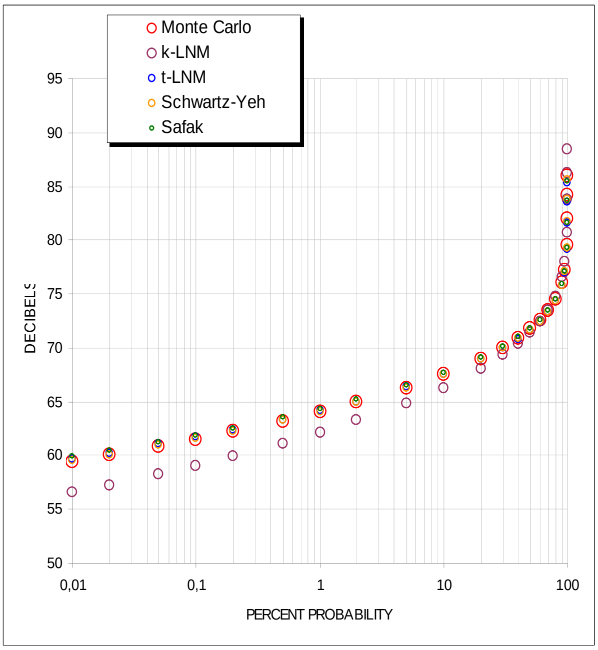

Figure 4. Results of the sum of the of four groups with this same standard deviations 7 dB:

- 6 RVs with m = 60 dB,
- 6 RVs with m = 50 dB,
- 6 RVs with m = 40 dB,
- 6 RVs with m = 30 dB

Sample of the radio network calculation have been made for case of DVB-T SFN with 13 transmitters. The signal parameters for the sample are given in Table 1.

Table 1. DVB-T SFN parameters

Parameter	Value
Mode	8 k
Guard interval	¼
Modulation	64-QAM
Coding	¾
Channel model	Rice
Reception	Fixed
Receiver antenna	Directional [1]
Channel / Frequency	36 / 594 MHz
Standard Deviation	5.5 dB

The following figures present the results of coverage areas of the SFN calculated according to different power sum methods. Table 2 shows percentages of coverage areas for required percentages of the time, compared to the whole allotment area.

Table 2. Comparisons of the calculated percentage of the allotment area covered for different expected coverage area probability for methods discussed.

	Coverage area probability [%]				
	50	70	90	95	99
k-LNM	100,0	99,5	95,1	91,2	81,0
t-LNM	100,0	99,5	95,4	91,5	81,1
Schwartz-Yeh	100,0	99,5	95,4	91,4	80,9
Analytical	100,0	99,6	95,7	91,8	81,6
Monte Carlo	100,0	99,6	95,5	91,6	81,2

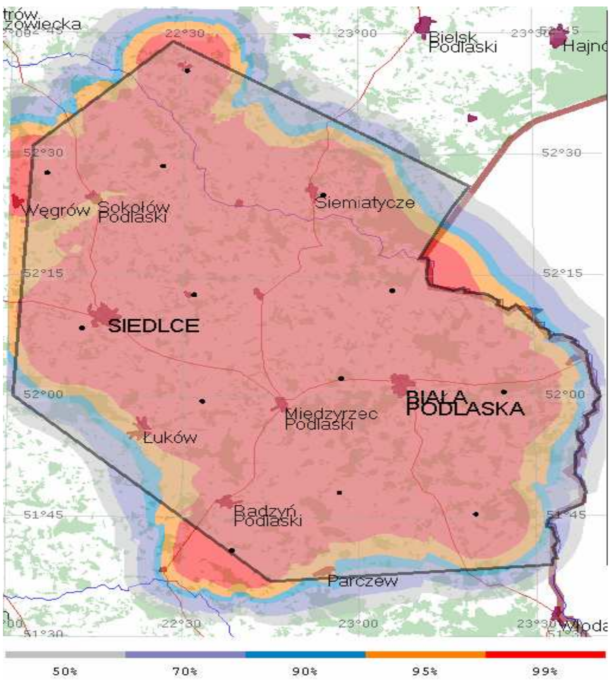

Figure 5. Coverage probability results using *k*-LNM7

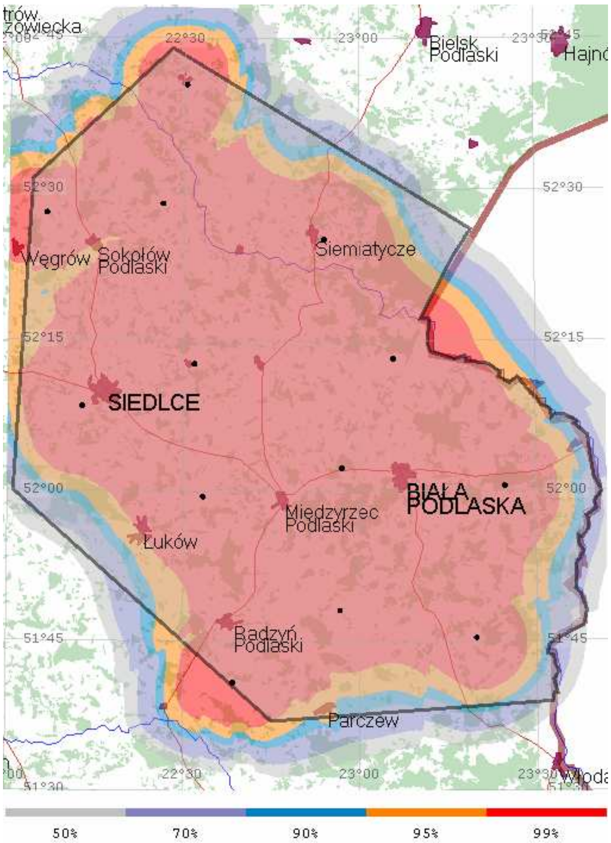

Figure 6. Coverage probability results - *t*-LNM

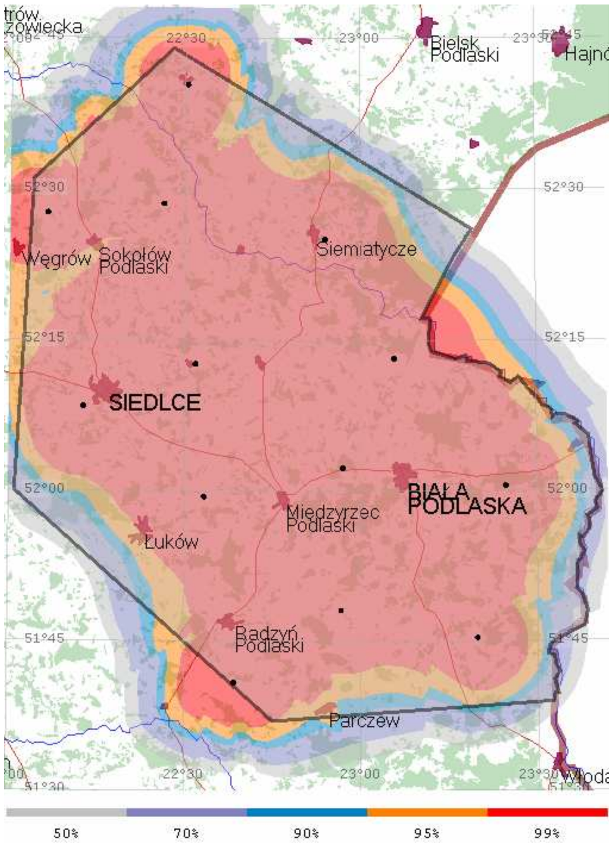

Figure 8. Coverage probability results – analytical/Safak

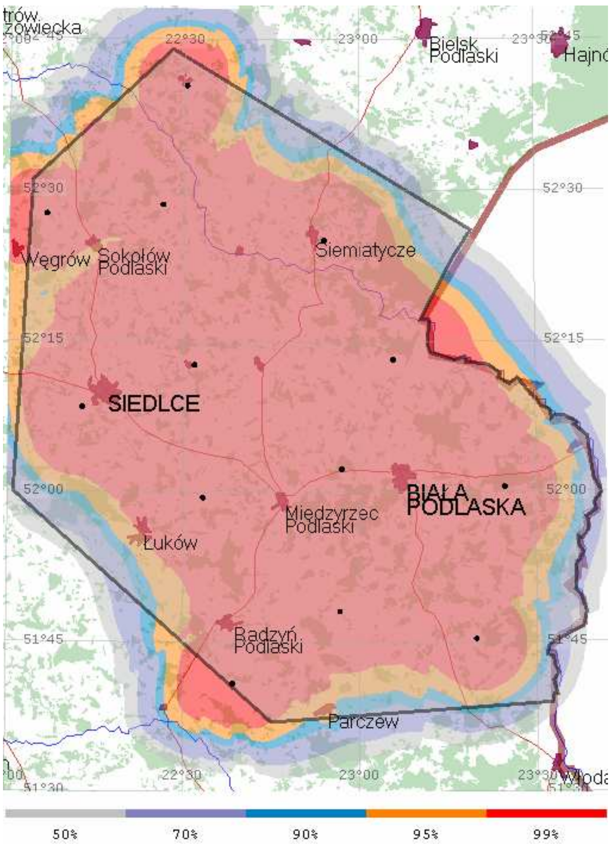

Figure 7. Coverage probability results - Schwartz-Yeh

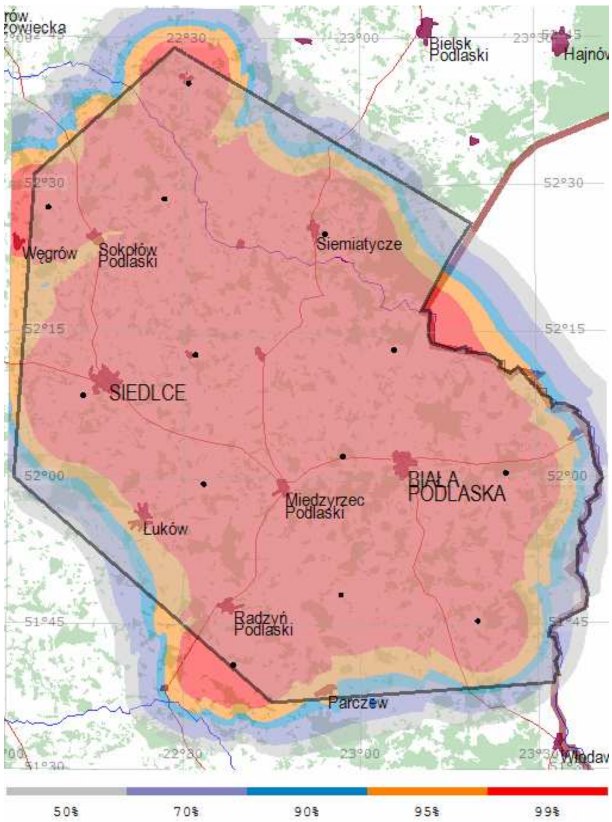

Figure 9. Coverage probability results – Monte Carlo

5. CONCLUSIONS

Precise radio network coverage calculations for planning purposes are very important nowadays especially if the electromagnetic compatibility of different systems should be guaranteed. More accurate results can be achieved using more sophisticated methods in the interference potential calculations. Today the computer systems can perform much more detailed calculations giving the very precise coverage maps for the interested network operators. In each analysis appropriate method should be chosen for each type of network and service. For example, as it was shown in the paper, the best method for the high probability calculations (90% or more) for summation of a few different broadcasting signals in the SFN network assuming self-interference calculation with 5.5dB standard deviation is the t-LNM method which was presented in this paper. However more precise calculations in many cases can increase precision only in few tenths of decibels. But in cases of automatic optimization of the radio network such slight results is wanted for better algorithms performance. The most accurate method (Monte Carlo) is a very time consuming so in practice it is no needed in standard network planning. But in some cases where there very precise calculations is sought e.g. security or aid/rescue services with many electromagnetic compatibility problems with other systems the very accurate method is desirable.

ACKNOWLEDGMENT

The authors would like to thank Mrs. Karina L. Beeke for her suggestion in numerical implementation of the analytical representation of the power sum.

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